Problem 11) Hermite's equation: $f''(x) - 2xf'(x) + 2nf(x) = 0$.

The method of Frobenius: $f(x) = \sum_{k=0}^{\infty} A_k x^{k+s} \Rightarrow f'(x) = \sum_{k=0}^{\infty} (k+s) A_k x^{k+s-1}$

$\Rightarrow f''(x) = \sum_{k=0}^{\infty} (k+s)(k+s-1) A_k x^{k+s-2}$. Substituting the above into Hermite's equation, we'll have:

$$\sum_{k=0}^{\infty} (k+s)(k+s-1) A_k x^{k+s-2} - 2\sum_{k=0}^{\infty} (k+s) A_k x^{k+s} + 2n \sum_{k=0}^{\infty} A_k x^{k+s} = 0$$

Let $k' = k-2$, then write the first sum as $s(s-1)A_0 x^{s-2} + s(s+1)A_1 x^{s-1} +$

$$\sum_{k=2}^{\infty} (k+s)(k+s-1) A_k x^{k+s-2} = s(s-1)A_0 x^{s-2} + s(s+1)A_1 x^{s-1} + \sum_{k'=0}^{\infty} (k'+2+s)(k'+s+1) A_{k'+2} x^{k'+s}.$$

Renaming the dummy variable $\underline{k'}$ once again as $\underline{k}$, Hermite's equation becomes:

$$s(s-1)A_0 x^{s-2} + s(s+1)A_1 x^{s-1} + \sum_{k=0}^{\infty} \left[(k+s+1)(k+s+2)A_{k+2} - 2(k+s-n)A_k\right] x^{k+s} = 0 \Rightarrow$$

$$\begin{cases} s(s-1)A_0 = 0, \\ s(s+1)A_1 = 0, \\ (k+s+1)(k+s+2)A_{k+2} = 2(k+s-n)A_k; \quad k \geqslant 0. \end{cases}$$

Case I) Let the solution to the indicial equations be $s=0$. Then both A_0 and A_1 will be arbitrary, and $A_{k+2} = \dfrac{2(k-n)}{(k+1)(k+2)} A_k$.

Case II) If the solution of the indicial equations is taken as $s=1$, then A_0 will be arbitrary, but $A_1 = 0$. The recursion relation then yields:

$A_3 = A_5 = A_7 = \dots = 0$ and $A_{k+2} = \dfrac{2(k+1-n)}{(k+2)(k+3)} A_k$ for $k = 0, 2, 4, 6, \dots$.

This is the same solution as obtained in Case I, except that k is shifted by 1.

Considering that, in the present case, $f(x) = \sum_{k=0}^{\infty} A_k x^{k+1} = \sum_{k=1}^{\infty} A_{k-1} x^k$, it is obvious that the solution obtained for $s=1$ is already embedded in the more general solution obtained for $s=0$.

Case III) Choosing $s=-1$ as the solution of the indicial equations makes A_1 arbitrary, but A_0 must then be set to zero. The recursion relation then ensures that $A_2 = A_4 = A_6 = \cdots = 0$. We'll have $A_{k+2} = \frac{2(k-n-1)}{k(k+1)} A_k$ for $k = 1, 3, 5, \ldots$. This is the same recursion relation as obtained in Case I, except that k is shifted by one unit. Considering that, in the present case, $f(x) = \sum_{k=1,3,\ldots}^{\infty} A_k x^{k-1} = \sum_{k'=0,2,4,\ldots}^{\infty} A_{k'+1} x^{k'}$, it is clear that this solution is also embedded in the more general solution obtained in the case of $s=0$.

We now return to the general solution of Case I, and evaluate the coefficients:

$$A_2 = -\frac{2n}{1\cdot 2} A_0 ; \quad A_4 = -\frac{2(n-2)}{3\cdot 4} A_2 = \frac{2^2 n(n-2)}{1\cdot 2\cdot 3\cdot 4} A_0 ;$$

$$A_6 = -\frac{2(n-4)}{5\cdot 6} A_4 = -\frac{2^3 n(n-2)(n-4)}{6!} A_0 ; \quad \cdots \Rightarrow A_{2m} = \frac{(-2)^m n(n-2)\cdots(n-2m+2)}{(2m)!} A_0 .$$

The series $\sum_{m=0}^{\infty} A_{2m} x^{2m}$ will terminate if n is an even integer, otherwise it will have an infinite number of terms.

$$A_3 = -\frac{2(n-1)}{2\cdot 3} A_1 ; \qquad A_5 = -\frac{2(n-3)}{4\cdot 5} A_3 = \frac{2^2 (n-1)(n-3)}{2\cdot 3\cdot 4\cdot 5} A_1 ; \quad \cdots$$

$$A_{2m+1} = \frac{(-2)^m (n-1)(n-3)\cdots(n-2m+1)}{(2m+1)!} A_1 .$$

The series $\sum_{m=0}^{\infty} A_{2m+1} x^{2m+1}$ will terminate if n is an odd integer, otherwise it will have an infinite number of terms.